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COMMENT

The $\bar{H}$ function associated with a certain class of Feynman integrals

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Abstract. Motivated by some further examples of the use of Feynman integrals which arise in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions, a generalization of the familiar H function of Fox was proposed recently by Inayat-Hussain. A brief discussion of contour selection and convergence conditions of the integrals involved is presented. It is also shown how some recent work of Buschman can be applied to derive various recurrence relations for the general $\bar{H}$ function. Finally, some relevant comments are made on the validity and novelty of two summation formulae for the Clausenian hypergeometric function, which were derived by Inayat-Hussain.

By evaluating (in two different ways) certain Feynman integrals which arise naturally in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions, Inayat-Hussain (1987a) derived a number of interesting characteristics of hypergeometric functions of one and more variables. More importantly, while presenting further examples of the use of these Feynman integrals (Inayat-Hussain 1987b), he was led to a new generalization of the familiar H function of Fox (1961). This general $\bar{H}$ function of Inayat-Hussain (1987b) contains the polylogarithm of a complex order and the exact partition function of the Gaussian model in statistical mechanics; indeed, in terms of a Mellin-Barnes contour integral, it is defined by

$$\bar{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}(\alpha_j, A_j; a_j)_{1,n}, (\alpha_j, A_j)_{n+1,p}\\(\beta_j, B_j)_{1,m}, (\beta_j, B_j; b_j)_{m+1,q}\end{matrix}\right.\right]=\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Theta(s)z^s\,\mathrm{d}s \tag{1}$$

where

$$\Theta(s)=\left[\prod_{j=1}^{m}\Gamma(\beta_j-B_js)\prod_{j=1}^{n}\{\Gamma(1-\alpha_j+A_js)\}^{a_j}\right]\times\left[\prod_{j=m+1}^{q}\{\Gamma(1-\beta_j+B_js)\}^{b_j}\prod_{j=n+1}^{p}\Gamma(\alpha_j-A_js)\right]^{-1} \tag{2}$$

which contains fractional powers of some of the Γ functions. Here, and in what follows, α_j $(j=1,\ldots,p)$ and β_j $(j=1,\ldots,q)$ are complex parameters,

$$A_j>0 \qquad (j=1,\ldots,p)$$
$$B_j>0 \qquad (j=1,\ldots,q) \tag{3}$$

and the exponents

$$a_j \qquad (j=1,\ldots,n) \qquad \text{and} \qquad b_j \qquad (j=m+1,\ldots,q)$$

can take on non-integer values.

Evidently, when the exponents a_j and b_j all take on integer values, the $\bar{H}$ function reduces to the familiar H function which has been studied and used in the literature rather extensively (see, e.g., Srivastava *et al* 1982).

Just as in the usual definition of the H function, the contour in (1) is presumed to be the imaginary axis:

$$\operatorname{Re}(s)=0 \tag{4}$$

which is suitably indented in order to avoid the singularities of the Γ functions and to keep those singularities on appropriate sides. This and other relevant restrictions on the parameters are the same as those given in Srivastava *et al* (1982).

No serious problem arises generally in regard to the location of singularities. For a_j not an integer, the poles of the Γ function of the numerator in (2) are converted to branch points. However, as long as there is no coincidence of poles from any $\Gamma(\beta_j - B_j s)$ and $\Gamma(1-\alpha_j + A_j s)$ pair, the branch cuts can be chosen so that the path of integration can be distorted in the usual manner. (Even if the poles interface on a horizontal line, the branch cuts can be chosen to allow this to happen.)

In connection with the convergence of the defining integral (1) for the $\bar{H}$ function, we recall from Erdélyi *et al* (1953) that

$$|\Gamma(x+\mathrm{i}y)| \sim (2\pi)^{1/2}|y|^{x-1/2}\,\mathrm{e}^{-\pi|y|/2} \qquad (|y|\to\infty) \tag{5}$$

along the lines $\operatorname{Re}(s)=x$. Making use of (5), we readily obtain the asymptotic for

$$|\{\Gamma(1-\alpha_j+A_j s)\}^{a_j}|.$$

Thus we need only slightly to modify the sufficient condition for absolute convergence of the contour integral (1) as follows:

$$\Omega \equiv \sum_{j=1}^{m}|B_j| + \sum_{j=1}^{n}|a_j A_j| - \sum_{j=m+1}^{q}|b_j B_j| - \sum_{j=n+1}^{p}|A_j| > 0. \tag{6}$$

This condition evidently provides exponential decay of the integrand in (1), and the region of (absolute) convergence in (1) is

$$|\arg(z)| < \tfrac{1}{2}\pi\Omega \tag{7}$$

where Ω is given by (6).

The criteria for convergence of the contour integral in (1) when $\Omega=0$ are more complicated (see, e.g., Dixon and Ferrar 1936; see also Erdélyi *et al* 1953, pp 49–50).

Recurrence relations for the G and H functions, and for some related functions, have appeared in the works of Buschman (1972, 1987, 1990). Contiguous function relations (that is, relations in which one parameter only is shifted by ± 1 in each of the terms) provide basic recurrences from which others can be derived. The results of Buschman (1990) would apply directly to the $\bar{H}$ function defined by (1). Constant coefficient contiguous function relations, which involve only parameters from those Γ functions with exponents equal to 1, are identical to those for the familiar H function. For example, if

$$\tilde{H} = \bar{H}^{1,1}_{2,2}\left[z \,\middle|\, \begin{matrix} (\alpha_1, A_1; 2), (\alpha_2, A_2) \\ (\beta_1, B_1), (\beta_2, B_2; \tfrac{3}{2}) \end{matrix} \right] \tag{8}$$

or, equivalently,

$$\tilde{H}=\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\Gamma(\beta_1-B_1s)\{\Gamma(1-\alpha_1+A_1s)\}^2}{\{\Gamma(1-\beta_2+B_2s)\}^{3/2}\Gamma(\alpha_2-A_2s)}z^s\,\mathrm{d}s \tag{9}$$

we have the contiguous function relation:

$$A_2\tilde{H}[\beta_1+1]-B_1\tilde{H}[\alpha_2-1]=\{\beta_1A_2-(\alpha_2-1)B_1\}\tilde{H}. \tag{10}$$

A recurrence relation involving the parameter α_1 can be obtained by the same methods, although not directly from the results given in the aforementioned papers. The squared Γ function in (9) forces us to consider a four-term recurrence relation of the form:

$$\lambda_1\tilde{H}[\alpha_1-1]+\lambda_2\tilde{H}[\beta_1+1]+\lambda_3\tilde{H}[\beta_1+2]-\lambda_0\tilde{H}=0 \tag{11}$$

which incidentally is not a contiguous function relation. By direct substitution, and upon sorting out the essential factors, we observe from (11) that

$$\lambda_1(1-\alpha_1+A_1s)^2+\lambda_2(\beta_1-B_1s)+\lambda_3(\beta_1+1-B_1s)(\beta_1-B_1s)-\lambda_0=0. \tag{12}$$

The squared factor in (12) illustrates the need for four terms and shows why the term $\tilde{H}[\beta_1+2]$ is important in (11).

Solving for the undetermined coefficients λ_1, λ_2, λ_3 and λ_0, we finally obtain the recurrence relation:

$$\begin{aligned}B_1^2\tilde{H}[\alpha_1-1]+A_1\{(2\beta_1+1)A_1-2(\alpha_1-1)B_1\}\tilde{H}[\beta_1+1]-A_1^2\tilde{H}[\beta_1+2]\\ =\{\beta_1A_1-(\alpha_1-1)B_1\}^2\tilde{H}.\end{aligned} \tag{13}$$

A similar recurrence relation involving α_1 and α_2, as well as other recurrence relations, can be obtained by combining our two basic results (10) and (13). Because of the fractional exponent, recurrence relations involving β_2 are not available, at least by these methods. Other integer exponents can, however, be treated in a manner analogous to the results presented here.

Further generalizations of the H function, which allow powers of $\Gamma(\beta_j-B_js)$ to appear also in (2), present similar problems. The difficulties of the corresponding contour in (1) can be overcome in the above manner, except that we shall have to keep the cuts together along certain curves.

It should be noted that such factors as $(s+\eta)^{-m-1}$, which appears in the first formula in Inayat-Hussain (1987b, p 4126), can be introduced by the application of an integral operator to the H function. Making use of the tables of Mellin transforms (Erdélyi *et al* 1954), including the convolution theorem, it is not difficult to obtain the fractional integral operator expression:

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\varphi(s)}{(s+\eta)^{m+1}}x^{-s}\,\mathrm{d}s=\frac{x^{\eta}}{\Gamma(m+1)}\int_{x}^{\infty}\left\{\log\left(\frac{\xi}{x}\right)\right\}^m\xi^{-\eta-1}H(\xi^{-1})\,\mathrm{d}\xi \tag{14}$$

where

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\varphi(s)x^s\,\mathrm{d}s=H(x). \tag{15}$$

Upon replacing x by x^{-1}, we can rewrite (14) in the form:

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\varphi(s)}{(s+\eta)^{m+1}}x^{s}\,\mathrm{d}s=\frac{x^{-\eta}}{\Gamma(m+1)}\int_{x^{-1}}^{\infty}\{\log(\xi x)\}^m\xi^{-\eta-1}H(\xi^{-1})\,\mathrm{d}\xi \tag{16}$$

where $\varphi(s)$ and $H(x)$ are related, as before, by (15). Now set $\xi = u^{-1}$ in (16), and we readily obtain the following fractional integral operator expression:

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\varphi(s)}{(s+\eta)^{m+1}}x^s\,ds = \frac{x^{-\eta}}{\Gamma(m+1)}\int_0^x\left\{\log\left(\frac{x}{u}\right)\right\}^m u^{\eta-1}H(u)\,du \qquad (17)$$

which provides us with an interesting alternative to the expression given by (14).

We note in passing that the value $d=6$ must be excluded from the interval of validity of the last formula in Inayat-Hussain (1987b, p 4128). Furthermore, the hypergeometric summation formula:

$${}_3F_2\left[\begin{matrix}1, 1, \frac{3}{2}; \\ 2, 2;\end{matrix}\; x\right] = \frac{4}{x}\ln\left(\frac{2}{1+(1-x)^{1/2}}\right) \qquad (0<|x|\leqslant 1) \qquad (18)$$

which was claimed to be a new result by Inayat-Hussain (1987b, p 4127), is actually a well known reduction formula for the Clausenian hypergeometric function (cf, e.g., Prudnikov *et al* 1986, p 519, entry 365; see also Nishimoto and Srivastava 1989, p 103, equation (3.9) for a rather elementary derivation).

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